7.1. Locally symmetric spaces. Let M be a connected m-dimensional Riemannian manifold. Then M is called *locally symmetric* if for all $p \in M$ there is a normal neighborhood $B(p,r)$ such that the *local geodesic reflection*

$$\sigma_p := \exp_p \circ (-\mathrm{id}) \circ \exp_p^{-1} : B(p,r) \to B(p,r)$$

is an isometry.

(a) Show that if M is locally symmetric, then $DR \equiv 0$.

Hint: Use that $d(\sigma_p)_p = -\mathrm{id}$ on T_pM.

(b) Suppose that $DR \equiv 0$. Show that if $c : [-1,1] \to M$ is a geodesic and $\{E_i\}_{i=1}^m$ is a parallel orthonormal frame along c, then

$$R(E_i, c')c' = \sum_{k=1}^{m} r_i^k E_k$$

for constants r_i^k.

(c) Show that if $DR \equiv 0$, then M is locally symmetric.

Hint: Let $q \in B(p,r)$, $q \neq p$, and $v \in T_qM$. To show that $|d(\sigma_p)_q(v)| = |v|$, consider the geodesic $c : [-1,1] \to B(p,r)$ with $c(0) = p$, $c(1) = q$, and a Jacobi field Y along c with $Y(0) = 0$, $Y(1) = v$. Use (b).

Solution. (a) Suppose M is locally symmetric. Let $p \in M$ and $w, x, y, z \in T_pM$. Since σ_p is an isometry and $d(\sigma_p)_p = -\mathrm{id}$, we have:

$$\begin{aligned}
-(D_wR)(x,y)z &= d(\sigma_p)_p((D_wR)(x,y)z) \\
&= D_{d(\sigma_p)_pw}(d(\sigma_p)_px, d(\sigma_p)_py)d(\sigma_p)_pz \\
&= (D_{-w}R)(-x,-y)(-z) \\
&= (D_wR)(x,y)z
\end{aligned}$$

Hence, $(D_wR)(x,y)z = 0$.

(b) Recall that for vector fields $X, Y, Z, W \in \Gamma(TM)$,

$$D_W(R(X,Y)Z) = R(X,Y)D_WZ + R(D_WX,Y)Z + R(X,D_WY)Z + (D_WR)(X,Y)Z.$$

Let $R(E_i, c')c' = \sum_{k=1}^m f_i^k E_k$ for functions $f_i^k : [-1,1] \to \mathbb{R}$. Since E_i and c' are parallel:

$$\begin{aligned} 0 = (D_{\frac{\partial}{\partial_t}} R)(E_i, c')c' &= D_{\frac{\partial}{\partial_t}}(R(E_i, c')c') \\ = \sum_{k=1}^m D_{\frac{\partial}{\partial_t}}(f_i^k E_k) &= \sum_{k=1}^m \dot{f}_i^k E_k + f_i^k D_{\frac{\partial}{\partial_t}} E_k \\ &= \sum_{k=1}^m \dot{f}_i^k E_k, \end{aligned}$$

which implies $\dot{f}_i^k = 0$ and therefore f_i^k is constant.

(c) Let $q \in B(p, r)$, $q \neq p$, and $v \in T_q M$. We must show that $|v| = |d(\sigma_p)_q(v)|$. Let $c : [-1, 1] \to M$ be the geodesic with $c(0) = p$, $c(1) = q$ and let Y be the Jacobi field along c with $Y(0) = 0$, $Y(1) = v$. Since σ_p reverts geodesics, $d(\sigma_p)_q(Y(1)) = Y(-1)$, and therefore we must show that $|Y(1)| = |Y(-1)|$. Write $Y = \sum_{i=1}^m h^i E_i$ for functions $h^i : [-1,1] \to \mathbb{R}$. The Jacobi equation implies:

$$\ddot{h}^k + \sum_{i=1}^m h^i r_i^k = 0$$

with $h^i(0) = 0$ for $i = 1, \ldots, m$. It follows that $h^i(-t) = -h^i(t)$ for all $t \in [-1, 1]$, and in particular $|Y(-1)| = |Y(1)|$.

□

7.2. Conjugate points in manifolds with curvature bounded from above.

(a) Prove directly, without using the Rauch Comparison Theorem, that there are no conjugate points in manifolds with non-positive sectional curvature.

(b) Show that in manifolds with sectional curvature at most κ, where $\kappa > 0$, there are no conjugate points along geodesics of length $< \pi/\sqrt{\kappa}$.

(c) Show that if $c : [0, \pi/\sqrt{\kappa}] \to M$ is a unit speed geodesic in a manifold with $\sec \geq \kappa > 0$, then some $c(t)$ is conjugate to $c(0)$ along $c|_{[0,t]}$.

Solution. (a) Let Y be a Jacobi field along a geodesic $c : [0, l] \to M$ with $Y(0) = 0$, and define $f : [0, l] \to \mathbb{R}$, $f(t) := |Y(t)|^2 \geq 0$. By our assumption, $R(Y, c', Y, c') \leq$

0 and therefore:

$$\begin{aligned} f'(t) &= 2\langle Y(t), Y'(t)\rangle; \\ f''(t) &= 2\langle Y'(t), Y'(t)\rangle + 2\langle Y(t), Y''(t)\rangle \\ &= 2|Y'(t)|^2 - 2R(Y, c'(t), Y, c'(t)) \geq 2|Y'(t)|^2 \geq 0. \end{aligned}$$

This implies that f is convex, therefore if $Y(t) = 0$ for some $t > 0$, then $f_{|_{[0,t]}} \equiv 0$, implying $Y \equiv 0$.

(b) First consider M_κ, the model space with constant curvature κ. Let $\overline{c} : [0, l] \to M_\kappa$ a geodesic with $|\overline{c}'(t)| = 1$ and $\overline{Y}$ a Jacobi field along $\overline{c}$ with $\overline{Y}(0) = 0$. Such a Jacobi field is given by:

$$\overline{Y}(t) = at\overline{c}'(t) + b\sin(\sqrt{\kappa}t)N(t)$$

where N is normal and parallel to $\overline{c}$. Then $|\overline{Y}(t)| > 0$ for $t \in (0, \pi/\sqrt{\kappa})$, $(a, b) \neq (0, 0)$, and therefore $\overline{c}(t)$ is not conjugate to $\overline{c}(0)$ along $\overline{c}$. For a manifold M with $\sec \leq \kappa$ we can now apply the Rauch comparison theorem for M and M_κ. We conclude that if Y is a Jacobi field with $Y(0) = 0$ and $Y'(0) \neq 0$ along some geodesic $c : [0, l] \to M$ with $L(c) \leq \pi/\sqrt{\kappa}$, we have $|Y(t)| \geq |\overline{Y}(t)| > 0$.

(c) Assume there are no conjugate points along c. Let $\overline{c} : [0, \pi/\sqrt{\kappa}] \in M_\kappa$ be a geodesic and $\overline{Y}(t) = \sin(\sqrt{\kappa}t)N(t)$ for some normal and parallel vector field N along $\overline{c}$. Let Y be a normal Jacobi field along c with $Y(0) = 0$ and $|Y'(0)| = |\overline{Y}'(0)|$. Then the Rauch comparison theorem implies $|\overline{Y}(\pi/\sqrt{\kappa})| \geq |Y(\pi/\sqrt{\kappa})| > 0$, a contradiction.

□

7.3. Volume comparison. Let M be an m-dimensional Riemannian manifold with sectional curvature $\sec \leq \kappa$, $p \in M$, and $r > 0$ such that $\exp_p|_{B_r(0)}$ is a diffeomorphism. Let $V_\kappa^m(r)$ denote the volume of a ball of radius r in the m-dimensional model space M_κ^m. Prove that:

$$V(B_r(p)) \geq V_\kappa^m(r)$$

Solution. If $\kappa > 0$, then $V_\kappa^m(r) = V_\kappa^m(D_\kappa)$ for $r > D_\kappa := \pi/\sqrt{\kappa}$. So we may assume $r \leq D_\kappa$. Choose a base point $\overline{p}$ in M_κ^m and a linear isometry $H : T_pM \to T_{\overline{p}}(M_\kappa^m)$. Recall from Proposition 1.21 that $\exp_p(B_r) = B_r(p)$ and let $F : B_r(p) \to B_r(p)$ be defined by:

$$F := \exp_{\overline{p}} \circ H \circ (\exp_p|_{B_r})^{-1}$$

The proof of Corollary 3.20 shows that for all $x, w \in T_pM$ with $|x| < r$ we have

$$|d(\exp_p)_x w| \geq |d(\exp_{\overline{p}})_{Hx} Hw|$$

Thus for all $q \in B_r(p)$ and $v \in T_qM$ we have $|dF_q(v)| \leq |v|$, so the volume distortion $J_F(q)$ of F at q is less than 1. Then:

$$V_\kappa^m(r) = V(B_{\overline{p}}(r)) = \int_{B_p(r)} J_F(q)\, dV(q) \leq V(B_p(r)).$$

$\square$